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ON THE ABSOLUTE ψ - SUMMABILITY FACTORS OF THE INFINITE SERIES

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ABSTRACT

In this paper is to prove a more general theorem of BOR [1] for absolute ψ - summability factor of the infinite series.

DEFINITIONS AND NOTATIONS

Let $A=(a_{n,k})$ be an infinite matrix of complex numbers $a_{nk}\,(nk=1,2,3,........)$ and let (ψ_n) be a sequence of complex numbers. Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By $A_n(s)$ we denote the A - transform of the sequence $s=(s_k)$, that is

$$A_n(s)=\sum_{k=1}^{\infty} a_{nk}s_k$$

The series $\sum a_n$ is said to be Summable $|A|$, if

$$\sum_{n=1}^{\infty}\left|A_n(s)-A_{n-1}(s)\right|<\infty$$

And it is said to be Summable $\psi-|A|_k\ k\geq 1$, if

$$\sum_{n=1}^{\infty}\left|\psi_n[A_n(s)-A_{n-1}(s)]\right|,\ k<\infty$$

If we take $\psi_n=n^{1-k^{-1}}\left(resp\ \psi_n=n^{\delta+1-k^{-1}},\ \delta\geq 0\right)$,

Then $\psi-|A|_k$ Summability is the same as $|A|_k\left(resp\ |A:\delta|_K\right)$ Summability.

INTRODUCTION

In 1965 Mishra [4] proved the following theorems :

THEOREM A : Let (λ_n) be a convex sequence such that $\sum \frac{\lambda_n}{n}$ is convergent, if $\sum a_n$ is bounded $|R, \log_n, 1|_k$, then $\sum a_n \lambda_n$ is Summable $|c,1|_k$ $k \geq 1$

Generalizing the above theorem MISHRA AND SRIVASTAVA [5] proved the following theorem.

THEOREM B : Let (χ_n) be a positive non – decreasing sequence and there be sequences (β_n) and (ε_n) such that

$$|\Delta\varepsilon_n| \leq \beta_n$$

$$\beta_n \to \infty \quad \text{as} \quad n \to \infty$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|\chi_n < \infty$$

$$|\varepsilon_n|\chi_n = o(1)$$

$$\text{If} \quad \sum_{v=1}^{n} \frac{|s_v|_k}{v} = o(\chi_n)$$

For $k \geq 1$ then $\sum a_n \varepsilon_n$ is Summable $|c,1|_k$. recently BOR [1] generalized the above theorem their theorem is as follows :

THEOREM C : Let (λ_n) be a positive non – decreasing sequence and the sequences (β_n) and (ε_n) are such that conditions

$$|\Delta\varepsilon_n| \leq \beta_n$$

$$\beta_n \to \infty \text{ as } n \to \infty$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|\chi_n < \infty$$

$$|\varepsilon_n|\chi_n = o(1)$$

Are satisfied . if there exists $\varepsilon > 0$ such that sequence $\left(n^{\varepsilon-k}|\psi_n|^k\right)$ is non- increasing and

$$\sum_{v=1}^{n} v^{\varepsilon-k}|\psi_v s_v|^k = o(\chi_n) \quad \text{as } n \to \infty$$

Then the series $\sum a_n\lambda_n$ is Summable $\psi-|c,1|_k$. $k \geq 1$

The object of this Paper is to prove a more general theorem than the above theorems. however, we shall prove the following theorem :

THEOREM : Let (χ_n) be a positive non – decreasing sequence and the sequences (β_n) and (λ_n) satisfy the following conditions.

$$|\Delta\lambda_n| \leq \beta_n \tag{1}$$

$$\beta_n \to \infty \text{ as } n \to \infty \tag{2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|\chi_n < \infty \tag{3}$$

$$|\lambda_n|\chi_n = o(1) \tag{4}$$

Moreover, if $\varepsilon > 0$ is such that the sequence $\left(n^{\varepsilon-k}|\psi_n|^k\right)$ is non- increasing and

$$\sum_{v=1}^{n} \frac{|\psi_v s_v|}{v} = o(1)\,(\chi_n\,\mu_n) \text{ as } n \to \infty \tag{5}$$

Where $\{\mu_n\}$ is positive non- increasing sequence and satisfies

$$n\chi_n\,\mu_n\Delta\left(\frac{1}{\mu_n}\right)=o(1) \quad \text{as} \quad n\to\infty \tag{6}$$

Then the series $\sum\frac{a_n\lambda_n}{\mu_n}$ is Summable $\psi-|c,1|$. It should be noted that our theorem also give results of BOR [1] for $k=1$

We need the following lemma for the proof of our theorem

Lemma 1 : Under the condition on (χ_n), (β_n) and (λ_n) as taken in the statement of the above theorem the following conditions hold, when (3) is satiesfied

$$n\beta_n\lambda_n=o(1) \tag{7}$$

And

$$\sum_{n=1}^{\infty}\beta_n\chi_n<\infty . \tag{8}$$

* PROOF OF THE THEOREMS *

Let u_n and t_n be with *Cesāro* means of order 1 of series $\sum a_n$ and of the sequence (n,a_n) respectively. Since $t_n=(u_n-u_{n-1})$ (see [2]) it is enough to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}|\psi_n\;T_n|<\infty \tag{9}$$

Where

$$T_n = -\frac{1}{n+1}\sum_{v=1}^{n}\frac{va_v\lambda_v}{\mu_v}$$

By Abel's transformation we get.

$$T_n = \frac{1}{n+1}\sum_{v=1}^{n-1}s_v\Delta\left(\frac{v\lambda_v}{\mu_v}\right)+\frac{1}{n+1}\frac{s_n n\lambda_n}{\mu_n}-\frac{s_0\lambda_1}{(n+1)\mu_1}$$

$$T_n = \frac{1}{n+1}\left[\sum_{v=1}^{n-1}s_v\frac{v\Delta\lambda_v}{\mu_v}+\frac{\lambda_{v+1}}{\mu+1}+v\lambda_{v+1}\Delta\left(\frac{1}{\mu}\right)\right]+\frac{1}{n+1}s_n\frac{n\lambda_n}{\mu_n}-\frac{1}{n+1}s_0\frac{\lambda_1}{\mu_1}$$

$$T_n = \frac{1}{n+1}\sum_{v=1}^{n-1}s_v\frac{v\Delta\lambda_v}{\mu_v}-\frac{1}{n+1}\sum_{v=1}^{n-1}s_v\frac{\lambda_{v+1}}{\mu_{v+1}}+\frac{1}{n+1}\sum_{v=1}^{n-1}s_v v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)+\frac{1}{n+1}s_n\frac{n\lambda_n}{\mu_n}-\frac{1}{n+1}s_0\frac{\lambda_1}{\mu_1}$$

$$T_n = T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}+T_{n,5} \qquad \text{(say)}$$

To complete the proof of the theorem, by Minkowski Inequality it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\psi_n\ T_{n,r}\right| < \infty \quad for\ r=1,2,3,4,5 \tag{10}$$

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n\ T_{n,1}\right| \le \sum_{n=2}^{m+1}\frac{|\psi_n|}{n^2}\left\{\sum_{v=1}^{n-1}\left|\frac{v\Delta\lambda_v}{\mu}\right||s_v|\right\}$$

$$= o(1)\sum_{v=1}^{m}\left|\frac{v\Delta\lambda_v}{\mu_v}\right||s_v\psi_v|\ v^{\varepsilon-1}\int_v^{\infty}\frac{1}{x^{\varepsilon+1}}dx$$

$$= o(1)\sum_{v=1}^{m}\left|\frac{v\Delta\lambda_v}{\mu_v}\right||s_v\psi_v|\ v^{-1}$$

$$\le o(1)\sum_{v=1}^{m}\frac{v\beta_v}{\mu_v}|s_v\psi_v|v^{-1} \tag{11}$$

In view of (1)

Applying partial summation to (11) that is to say, we have

$$\sum_{v=1}^{n}\frac{v\beta_v}{\mu_v}\left|\frac{\psi_v s_v}{v}\right| = o(1)\sum_{v=1}^{m-1}\Delta\left(\frac{v\beta_v}{\mu_v}\right)\sum_{r=1}^{v}\frac{|\psi_v s_r|}{r}+\frac{m\beta_m}{\mu_m}\sum_{v=1}^{m}\frac{|\psi_v s_v|}{v}$$

$$=o(1)\sum_{v=1}^{m-1}\left\{\Delta^2\frac{v\beta_v}{\mu_v}+\frac{\Delta\beta_v}{\mu_v}+\Delta\beta_v v\Delta\left(\frac{1}{\mu_v}\right)\right\}\sum_{r=1}^{v}\frac{|\psi_r s_r|}{r}$$

$$+\frac{m\beta_m}{\mu_m}\sum_{v=1}^{m}\frac{|\psi_v s_v|}{v}$$

$$=o(1)\sum_{v=1}^{m-1}\frac{\Delta^2\beta_v v}{\mu_v}\sum_{r=1}^{v}\frac{|\psi_r s_r|}{r}+o(1)\sum_{v=1}^{m-1}\frac{\Delta\beta_v}{\mu_v}\sum_{r=1}^{v}\frac{|\psi_r s_r|}{r}$$

$$+o(1)\sum_{v=1}^{m-1}\Delta\beta_v v\Delta\left(\frac{1}{\mu_v}\right)\sum_{r=1}^{v}\frac{|\psi_r s_r|}{r}+\frac{m\beta_m}{\mu_m}\sum_{v=1}^{m}\frac{|\psi_v s_v|}{v}=o(1)\sum_{v=1}^{m-1}\frac{\Delta^2\beta_v v}{\mu_v}\chi_v\mu_v+o(1)\sum_{v=1}^{m-1}\frac{\Delta\beta_v v}{\mu_v}\chi_v\mu_v$$

$$+o(1)\sum_{v=1}^{m-1}\Delta\beta_v v\Delta\left(\frac{1}{\mu_v}\right)\chi_v\mu_v+\frac{m\beta_m}{\mu_m}\chi_m\mu_m$$

$$=o(1)\sum_{v=1}^{m-1}\Delta^2\beta_v.v+o(1)\sum_{v=1}^{m-1}\Delta\beta_v\chi_v\mu_v+o(1)\sum_{v=1}^{m-1}\Delta\beta_v v+m\beta_m\chi_m$$

$$\sum_{v=1}^{n}\frac{v\beta_v}{\mu_v}\left|\frac{\psi_v s_v}{v}\right|=o(1)+o(1)+o(1)+o(1)$$

In view of (3), (4), (6), (7) and (8)

Hence

$$\sum_{n=2}^{m+1}\frac{1}{n}|\psi_n T_{n,1}|=o(1)\qquad as\ \ m\to\infty$$

Again

$$\sum_{n=2}^{m+1}\frac{1}{n}|\psi_n T_{n,2}|\le\sum_{n=2}^{m+1}\frac{1}{n^2}|\psi_n|\left\{\sum_{v=1}^{n-1}|s_v|\left|\frac{\lambda_{v+1}}{\mu_{v+1}}\right|\right\}$$

$$=\sum_{v=1}^{m}\left|\frac{\lambda_{v+1}}{\mu_{v+1}}\right||s_v|\sum_{n=v+1}^{m+1}\frac{|\psi_n|}{n^2}$$

$$=o(1)\sum_{v=1}^{m}\left|\frac{\lambda_{v+1}}{\mu_{v+1}}\right||s_v\psi_v|v^{\varepsilon-1}\sum_{n=v+1}^{m+1}\frac{1}{n^{\varepsilon+1}}$$

$$=o(1)\sum_{v=1}^{m}\left|\frac{\lambda_{v+1}}{\mu_{v+1}}\right||s_v\psi_v|v^{\varepsilon-1}\int_v^{\infty}\frac{1}{n^{\varepsilon+1}}dx$$

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$$= o(1)\sum_{v=1}^{m}\left|\frac{\lambda_{v+1}}{\mu_{v+1}}\right|\left|s_v\psi_v\right|v^{-1} \qquad (12)$$

Applying partial summation to (12). this is to say we have

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,2}\right| = o(1)\sum_{v=1}^{m-1}\Delta\left|\frac{\lambda_v}{\mu_{v+1}}\right|\left|\sum_{r=1}^{m}\left|s_r\psi_r\right|\right| + \frac{\lambda_m}{\mu_{m+1}}\sum_{v=1}^{m}\frac{\left|\psi_v s_v\right|}{v}$$

$$= o(1)\sum_{v=1}^{m-1}\left\{\Delta\frac{\lambda_v}{\mu_{v+1}} + \lambda_v\Delta\left(\frac{1}{\mu_{v+1}}\right)\right\}\sum_{r=1}^{v}\left|s_r\psi_r\right|\frac{1}{r} + \frac{\lambda_m}{\mu_{m+1}}\sum_{v=1}^{m}\frac{\left|\psi_v s_v\right|}{v}$$

$$= o(1)\sum_{v=1}^{m-1}\Delta\frac{\lambda_v}{\mu_{v+1}}\sum_{r=1}^{m}\frac{\left|\psi_r s_r\right|}{r} + o(1)\sum_{v=1}^{m-1}\lambda_v\Delta\left(\frac{1}{\mu_{v+1}}\right)\sum_{r=1}^{v}\left|s_r\psi_r\right|\frac{1}{r}$$
$$+\frac{\lambda_m}{\mu_{m+1}}\sum_{v=1}^{m}\frac{\left|\psi_v s_v\right|}{v}$$

$$= o(1)\sum_{v=1}^{m-1}\Delta\frac{\lambda_v}{\mu_{v+1}}\chi_v\mu_v + o(1)\sum_{v=1}^{m-1}\lambda_v\Delta\left(\frac{1}{\mu_{v+1}}\right)\chi_v\mu_v + \frac{\lambda_m}{\mu_{m+1}}\chi_m\mu_m$$

$$= o(1)\sum_{v=1}^{m-1}\chi_v\Delta\lambda_v + o(1)\sum_{v=1}^{m-1}\mu_v\chi_v\Delta\left(\frac{1}{\mu_{v+1}}\right) + \lambda_m\mu_m$$

$$= o(1)\sum_{v=1}^{m-1}\chi_v\lambda_v + o(1)\sum_{v=1}^{m-1}\lambda_v + o(1)$$

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,2}\right| = o(1) + o(1) + o(1)$$

In view of (1), (7), (11), (12) and (8).

Hence

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,2}\right| = o(1) \qquad as \qquad m\to\infty$$

Again

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,3}\right| \le \sum_{n=2}^{m+1}\frac{\left|\psi_n\right|}{n^2}\left\{\sum_{v=1}^{n-1}\left|s_v\right|v\left|\lambda_v\right|\Delta\left(\frac{1}{\mu_v}\right)\right\}$$

$$=o(1)\sum_{v=1}^{n-1}|s_v|v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)\sum_{n=v+1}^{m+1}\frac{|\psi_n|}{n^2}$$

$$=o(1)\sum_{v=1}^{n-1}|s_v|v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)|\psi_v|v^{\varepsilon-1}\sum_{n=v+1}^{m+1}\frac{1}{n^{\varepsilon+1}}$$

$$=o(1)\sum_{v=1}^{n-1}|s_v\psi_v|v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)v^{\varepsilon-1}\int_v^{\infty}\frac{1}{n^{\varepsilon+1}}dx$$

$$\Rightarrow \quad \sum_{n=2}^{m+1}\frac{1}{n}|\psi_n T_{n,3}| = \sum_{v=1}^{n-1}|s_v\psi_v|v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)v^{-1} \tag{13}$$

Applying partial summation to (13). that is to say we have

$$\sum_{n=2}^{m+1}\frac{1}{n}|\psi_n T_{n,3}|=o(1)\sum_{v=1}^{n-1}\Delta\left\{v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)\right\}\sum_{r=1}^{v}\frac{|s_r\psi_r|}{r}+m\lambda_m\Delta\left(\frac{1}{\mu_m}\right)\sum_{r=1}^{m}\frac{|s_r\psi_r|}{r}$$

$$\sum_{n=2}^{m+1}\frac{1}{n}|\psi_n T_{n,3}|=o(1)\sum_{v=1}^{n-1}\left\{|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)+v|\Delta\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)+v|\lambda_{v+1}|\Delta^2\left(\frac{1}{\mu}\right)\right\}\sum_{r=1}^{v}\frac{|s_r\psi_r|}{r}$$

$$+m|\lambda_m|\Delta\left(\frac{1}{\mu_v}\right)\sum_{r=1}^{m}\frac{|s_r\psi_r|}{r}$$

$$=o(1)\sum_{v=1}^{n-1}\left\{|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)+v|\Delta\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)+v|\lambda_{v+1}|\Delta^2\left(\frac{1}{\mu}\right)\right\}\chi_v\mu_v$$

$$+m|\lambda_m|\Delta\left(\frac{1}{\mu_m}\right)\chi_m\mu_m$$

$$=o(1)\sum_{v=1}^{n-1}|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)\chi_v\mu_v+o(1)\sum_{v=1}^{n-1}v|\Delta\lambda_v|\,\Delta\left(\frac{1}{\mu_v}\right)\chi_v\mu_v$$

$$+o(1)\sum_{v=1}^{n-1}v|\lambda_{v+1}|\,\Delta^2\left(\frac{1}{\mu_v}\right)\chi_v\mu_v+m|\lambda_{m+1}|\Delta\left(\frac{1}{\mu_m}\right)\chi_m\mu_m$$

$$=o(1)\sum_{v=1}^{n-1}\frac{|\lambda_v|}{v}+o(1)\sum_{v=1}^{n-1}|\Delta\lambda_v|+o(1)\sum_{v=1}^{n-1}|\lambda_{v+1}|+o(1)$$

$$\Rightarrow \quad \sum_{n=2}^{m+1}\frac{1}{n}|\psi_n T_{n,3}|=o(1)+\ o(1)+\ o(1)+\ o(1)$$

In view of (1), (2), (3) and (6)

Hence

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,3}\right| = o(1) \quad as \quad m \to \infty$$

Again also as in $T_{n,2}$ we have

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,4}\right| = o(1)\sum_{n=1}^{m}\frac{|\psi_n|}{\mu_n}\frac{|s_n\psi_n|}{n}$$

$$= o(1) \quad as \quad m \to \infty$$

Finally we have

$$\sum_{n=2}^{m+1}\frac{1}{n}\left|\psi_n T_{n,5}\right| = o(1)\sum_{n=1}^{m}\frac{|\psi_n|}{n^2}$$

$$= o(1)\sum_{n=1}^{m}\frac{n^{\varepsilon-1}|\psi_n|}{n^{\varepsilon+1}}$$

Since $\left(n^{\varepsilon-1}|\psi_n|\right)$ is non – increasing by hypothesis we have

Therefore we get

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\psi_n, T_n\right| < \infty$$

***This completes the proof of theorem**.*

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